

# Pseudocodewords from Bethe Permanents

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**Abstract**—It was recently conjectured that a vector with components equal to the Bethe permanent of certain submatrices of a parity-check matrix is a pseudocodeword. In this paper we prove a stronger version of this conjecture and investigate the families of pseudocodewords obtained by using the Bethe permanent in two particular cases. We also highlight some interesting properties of the permanent of block matrices and their effects on pseudocodewords.

**Index Terms**—Bethe permanents, permanents, low-density parity-check codes, pseudocodewords.

## I. INTRODUCTION

IN [1], a simple technique is presented for upper bounding the minimum Hamming distance of a binary linear code that is described by an  $m \times n$  parity-check matrix  $\mathbf{H}$ , based on explicitly constructing codewords with components equal to  $\mathbb{F}_2$ -determinants of some  $m \times m$  submatrices of  $\mathbf{H}$ . Subsequently, this technique was extended and refined in [2]–[4] in the case of quasi-cyclic binary linear codes and by computing those determinant components over the ring of integers  $\mathbb{Z}$  instead of over the binary field  $\mathbb{F}_2$  (and taking their absolute-value) and showing that the resulting integer vectors are pseudocodewords, i.e., vectors that lie in the fundamental cone of the parity-check matrix of the code [5], [6]. In addition, in [4], a closely related class of pseudocodewords called *perm-pseudocodewords* was defined, obtained by taking the vector components to be equal to the  $\mathbb{Z}$ -permanent of some  $m \times m$  submatrices of  $\mathbf{H}$  instead of the determinant.

Related to this last construction, Vontobel introduced in [7, Sec. IX] a similar vector but having components equal to the Bethe permanent of some  $m \times m$  submatrices of a matrix  $\mathbf{H}$  instead of the regular permanent, and conjectured that this vector is a pseudocodeword. The *Bethe permanent* was a term introduced in [7] to denote the approximation of a permanent of a non-negative matrix, i.e., of a matrix containing only non-negative real entries, by solving a certain Bethe free energy minimization problem. Vontobel also provided some reasons why the approximation works well, by showing that the Bethe free energy is a convex function and that the sum-product algorithm finds its minimum efficiently.

In this paper we state and prove a stronger version of the above mentioned conjecture and discuss some interesting properties of Bethe permanents and their relationships to pseudocodewords.

The remainder of the paper is structured as follows. In Section II we list basic notations and definitions, provide the necessary background, and formally define the class of perm-pseudocodewords and Bethe perm-vectors. In Section III we

give the main steps for proving the conjecture, and in Section IV we compute exactly the Bethe perm-pseudocodewords in two cases and offer some remarks and examples. We conclude the paper in Section V. In the Appendix we included a proof of an important lemma used to prove the conjecture and three lemmas used in Section IV.

The literature on permanents and adjacent areas of counting perfect matchings, counting 0-1 matrices with specified row and column sums, etc., is vast. Most relevant to this paper are the very recent work [8] that studies the so-called fractional free energy functionals and resulting lower and upper bounds on the permanent of a non-negative matrix, the paper [9] on counting matchings in graphs with the help of the sum-product algorithm, and the papers on max-product/min-sum algorithms based approaches to the maximum weight perfect matching problem [10]–[13].

## II. DEFINITIONS AND BACKGROUND

Let  $\mathbb{Z}$ ,  $\mathbb{R}$ , and  $\mathbb{F}_2$  be the ring of integers, the field of real numbers, and the finite field of size 2, respectively. Rows and columns of matrices and entries of vectors will be indexed starting at 1. If  $\mathbf{H}$  is some matrix and if  $\alpha = \{i_1, \dots, i_r\}$  and  $\beta = \{j_1, \dots, j_s\}$  are subsets of the row and column index sets, respectively, then  $\mathbf{H}_{\alpha, \beta}$  is the sub-matrix of  $\mathbf{H}$  that contains only the rows of  $\mathbf{H}$  whose index appears in the set  $\alpha$  and only the columns of  $\mathbf{H}$  whose index appears in the set  $\beta$ . If  $\alpha$  is the set of all row indices of  $\mathbf{H}$ , we will simply write  $\mathbf{H}_\beta$  instead of  $\mathbf{H}_{\alpha, \beta}$ . Moreover, for any set of indices  $\gamma$ , we will use the short-hand  $\gamma \setminus i$  for  $\gamma \setminus \{i\}$ . For an integer  $M$ , we will use the common notation  $[M] \triangleq \{1, \dots, M\}$ . For a set  $\alpha$ ,  $|\alpha|$  will denote the cardinality of  $\alpha$  (the number of elements in the set  $\alpha$ ). The set of all  $M \times M$  permutation matrices will be denoted by  $\mathcal{P}_M$ .

**Definition 1.** Let  $\theta = (\theta_{ij})$  be an  $m \times m$ -matrix over some commutative ring. Its determinant and permanent, respectively, are defined to be

$$\det(\theta) \triangleq \sum_{\sigma} \text{sgn}(\sigma) \prod_{i \in [m]} \theta_{i, \sigma(i)},$$

$$\text{perm}(\theta) \triangleq \sum_{\sigma} \prod_{i \in [m]} \theta_{i, \sigma(i)},$$

where the summation is over all  $m!$  permutations of the set  $[m]$ , and where  $\text{sgn}(\sigma)$  is the signature operator.  $\square$

In this paper, we consider only permanents over the integers.

**Definition 2.** Let  $\mathbf{H} = (h_{ij})$  be an  $m \times n$  parity-check matrix of some binary linear code. The fundamental cone  $\mathcal{K}(\mathbf{H})$  of

$\mathbf{H}$  is the set of all vectors  $\omega = (\omega_i) \in \mathbb{R}^n$  that satisfy

$$\omega_j \geq 0 \quad \text{for all } j \in [n], \quad (1)$$

$$\omega_j \leq \sum_{j' \in \text{supp}(\mathbf{R}_i) \setminus j} \omega_{j'} \quad \text{for all } i \in [m] \text{ and } j \in \text{supp}(\mathbf{R}_i), \quad (2)$$

where  $\mathbf{R}_i$  is the  $i$ -th row vector of  $\mathbf{H}$  and  $\text{supp}(\mathbf{R}_i)$  is its support (the positions where the vector is non-zero). A vector  $\omega \in \mathcal{K}(\mathbf{H})$  is called a pseudocodeword [14]. Two pseudo-codewords  $\omega, \omega' \in \mathcal{K}(\mathbf{H})$  are said to be in the same equivalence class if there exists an  $\alpha > 0$  such that  $\omega = \alpha \cdot \omega'$ . In this case, we write  $\omega \propto \omega'$ .  $\square$

There are two ways of characterizing pseudocodewords in a code  $\mathcal{C}$ : as codewords in codes associated to covers of the Tanner graph corresponding to  $\mathcal{C}$ , or using the computationally easier linear programming (LP) approach, which connects the presence of pseudocodewords in message passing iterative decoding and LP decoding [14], [15]. Here we take the latter approach.

**Definition 3.** Let  $\mathcal{C}$  be a binary linear code described by a parity-check matrix  $\mathbf{H} \in \mathbb{F}_2^{m \times n}$ ,  $m < n$ . For a size- $(m+1)$  subset  $\beta$  of  $[n]$  we define the perm-vector based on  $\beta$  to be the vector  $\omega \in \mathbb{Z}^n$  with components

$$\omega_i \triangleq \begin{cases} \text{perm}_{\mathbb{Z}}(\mathbf{H}_{\beta \setminus i}) & \text{if } i \in \beta \\ 0 & \text{otherwise} \end{cases}. \quad \square$$

In [4] it was shown that these vectors are in fact pseudocodewords. We state this here for easy reference together with its proof.

**Theorem 4.** (from [4]) Let  $\mathcal{C}$  be a binary linear code described by the parity-check matrix  $\mathbf{H} \in \mathbb{F}_2^{m \times n}$ ,  $m < n$ , and let  $\beta$  be a size- $(m+1)$  subset of  $[n]$ . The perm-vector  $\omega$  based on  $\beta$  is a pseudocodeword of  $\mathbf{H}$ .

*Proof:* We need to verify (1) and (2). From Definition 3 it is clear that  $\omega$  satisfies (1). To show that  $\omega$  satisfies (2), we fix an  $i \in [m]$  and a  $j \in \text{supp}(\mathbf{R}_i)$ . If  $j \notin \beta$  then  $\omega_j = 0$  and (2) is clearly satisfied. If  $j \in \beta$ , then

$$\begin{aligned} \sum_{j' \in \text{supp}(\mathbf{R}_i) \setminus j} \omega_{j'} &= \sum_{j' \in \text{supp}(\mathbf{R}_i) \setminus j} h_{i,j'} \omega_{j'} \\ &= \sum_{j' \in \beta \setminus j} h_{i,j'} \cdot \text{perm}(\mathbf{H}_{\beta \setminus j'}) + \sum_{j' \in ([n] \setminus \beta) \setminus j} h_{i,j'} \cdot 0 \\ &= \sum_{j' \in \beta \setminus j} h_{i,j'} \sum_{j'' \in \beta \setminus j'} h_{i,j''} \text{perm}(\mathbf{H}_{[m] \setminus i, \beta \setminus \{j', j''\}}) \\ &\stackrel{(*)}{\geq} \sum_{j' \in \beta \setminus j} h_{i,j'} h_{i,j} \text{perm}(\mathbf{H}_{[m] \setminus i, \beta \setminus \{j', j\}}) \\ &= h_{i,j} \sum_{j' \in \beta \setminus j} h_{i,j'} \text{perm}(\mathbf{H}_{[m] \setminus i, \beta \setminus \{j', j\}}) \\ &= h_{i,j} \text{perm}(\mathbf{H}_{\beta \setminus j}) = h_{i,j} \omega_j \stackrel{(**)}{=} \omega_j, \end{aligned}$$

where at step (\*) we kept only the terms for which  $j'' = j$ , and where step (\*\*) follows from  $j \in \text{supp}(\mathbf{R}_i)$ . Because  $i \in [m]$  and  $j \in \text{supp}(\mathbf{R}_i)$  were taken arbitrary, it follows that  $\omega$  satisfies (2).  $\blacksquare$

**Example 5.** Consider the  $[4, 2, 2]$  binary linear code  $\mathcal{C}$  based on the parity-check matrix  $\mathbf{H} \triangleq \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ , where  $n = 4$  and  $m = 2$ . The following list contains the perm-vectors based on all possible subsets  $\beta \subset [4]$  of size  $m+1 = 3$ :  $(2, 1, 1, 0)$ ,  $(1, 1, 0, 1)$ ,  $(1, 0, 1, 1)$ ,  $(0, 1, 1, 2)$ . It can be easily checked that these satisfy the inequalities of the fundamental cone above, as the theorem predicts.  $\square$

The following combinatorial description of the Bethe permanent can be found in [7]. We use it here as a definition.

**Definition 6.** Let  $\theta$  be a non-negative (with non-negative real entries)  $m \times m$  matrix and  $M$  be a positive integer. Let  $\Psi_{m,n,M}$  be the set

$$\Psi_{m,n,M} \triangleq \mathcal{P}_M^{m \times n} = \{\mathbf{P} = (P_{ij})_{\substack{i \in [m] \\ j \in [n]}} \mid P_{ij} \in \mathcal{P}_M\}.$$

If  $m = n$ , we will use  $\Psi_{m,M} \triangleq \Psi_{m,n,M}$ .

For a matrix  $\mathbf{P} \in \Psi_{m,M}$ , the  $\mathbf{P}$ -lifting of  $\theta$  is defined as the  $mM \times mM$  matrix

$$\theta^{\uparrow \mathbf{P}} \triangleq \begin{pmatrix} \theta_{11} P_{11} & \dots & \theta_{1m} P_{1m} \\ \vdots & & \vdots \\ \theta_{m1} P_{m1} & \dots & \theta_{mm} P_{mm} \end{pmatrix},$$

and the degree- $M$  Bethe permanent of  $\theta$  is defined as

$$\text{perm}_{B,M}(\theta) \triangleq \sqrt[M]{\langle \text{perm}(\theta^{\uparrow \mathbf{P}}) \rangle_{\mathbf{P} \in \Psi_{m,M}}},$$

where the angular brackets represent the arithmetic average of  $\text{perm}(\theta^{\uparrow \mathbf{P}})$  over all  $\mathbf{P} \in \Psi_{m,M}$ .

Then, the Bethe permanent of  $\theta$  is defined as

$$\text{perm}_B(\theta) \triangleq \limsup_{M \rightarrow \infty} \text{perm}_{B,M}(\theta). \quad \square$$

**Remark 7.** Note that a  $\mathbf{P}$ -lifting of a matrix  $\theta$  corresponds to an  $M$ -cover graph of the protograph described by  $\theta$ , therefore we can consider  $\theta^{\uparrow \mathbf{P}}$  to represent a protograph-based LDPC code and  $\theta$  to be its protomatrix [16].  $\square$

**Definition 8.** Let  $\mathcal{C}$  be a binary linear code described by a parity-check matrix  $\mathbf{H} \in \mathbb{F}_2^{m \times n}$ ,  $m < n$ . For a size- $(m+1)$  subset  $\beta$  of  $[n]$  we define the Bethe perm-vector based on  $\beta$  to be the vector  $\omega_B \in \mathbb{R}^n$  with components

$$\omega_{B,i} \triangleq \begin{cases} \text{perm}_B(\mathbf{H}_{\beta \setminus i}) & \text{if } i \in \beta \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, we define degree- $M$  Bethe perm-vector based on  $\beta$  to be the vector  $\omega_{B,M} \in \mathbb{Q}^n$  with components

$$\omega_{B,M,i} \triangleq \begin{cases} \text{perm}_{B,M}(\mathbf{H}_{\beta \setminus i}) & \text{if } i \in \beta \\ 0 & \text{otherwise} \end{cases}. \quad \square$$

The following conjecture is stated in [7].

**Conjecture 9** ([7]). Let  $\mathcal{C}$  be a binary linear code described by an  $m \times n$  binary parity-check matrix  $\mathbf{H}$ , with  $m < n$ , and let  $\beta$  be a size- $(m+1)$  subset of  $[n]$ . Then the Bethe perm-vector  $\omega_B$  based on  $\beta$  is a pseudocodeword of  $\mathbf{H}$ , i.e.,  $\omega_B \in \mathcal{K}(\mathbf{H})$ .

We will in fact prove the following stronger form of the conjecture.

**Theorem 10.** Let  $\mathcal{C}$  be a binary linear code described by an  $m \times n$  binary parity-check matrix  $\mathbf{H}$ , with  $m < n$ , and let  $\beta$  be a size- $(m+1)$  subset of  $[n]$ . Then, for all  $M \geq 1$ , the degree- $M$  Bethe perm-vectors  $\omega_{B,M}$  and the Bethe perm-vector  $\omega_B$  based on  $\beta$  are pseudocodewords of  $\mathbf{H}$ , i.e.,  $\omega_{B,M} \in \mathcal{K}(\mathbf{H})$  and  $\omega_B \in \mathcal{K}(\mathbf{H})$ .

We will call these pseudocodewords the *degree- $M$  Bethe perm-pseudocodeword* based on  $\beta$  and the *Bethe perm-pseudocodeword* based on  $\beta$ , respectively, and, when we are considering both sets, we will call them the *Bethe pseudocodewords*.

In proving Theorem 10, it is enough to show that  $\omega_{B,M} \in \mathcal{K}(\mathbf{H})$ , for all  $M \geq 1$ . Then, by taking the limit it follows that  $\omega_B \in \mathcal{K}(\mathbf{H})$ , as the next lemma shows.

**Lemma 11.** Let  $\mathcal{C}$  and  $\beta$  as in Theorem 10. If  $\omega_{B,M} \in \mathcal{K}(\mathbf{H})$ , for all  $M \geq 1$  integers, then  $\omega_B \in \mathcal{K}(\mathbf{H})$ .

*Proof:* Since  $\omega_{B,M} \in \mathcal{K}(\mathbf{H})$ , for all  $M \geq 1$ , each of the inequalities in (1) and (2) is satisfied by  $\omega_{B,M,i}$ ,  $i \in [n]$ . Taking the limit when  $M \rightarrow \infty$  gives that  $\omega_{B,i}$  must also satisfy the same inequalities. It follows that  $\omega_B \in \mathcal{K}(\mathbf{H})$ . ■

### III. PROOF OF THEOREM 10

Before proving this theorem, we need a few lemmas and theorems.

Let  $\theta = \mathbf{1}_{m \times m}$  be the all-one matrix of size  $m \times m$ . Let  $I$  denote the identity matrix of size  $M \times M$  and for all  $m \geq 1$ , let

$$p_{m-1,m,M} \triangleq \sum_{\mathbf{Q} \in \Psi_{m-1,m,M}} \text{perm} \begin{bmatrix} I & I & \cdots & I \\ Q_{11} & Q_{12} & \cdots & Q_{1,m} \\ \vdots & \vdots & \cdots & \vdots \\ Q_{m-1,1} & Q_{m-1,2} & \cdots & Q_{m-1,m} \end{bmatrix},$$

$$q_{m-1,m,M} \triangleq \frac{p_{m-1,m,M}}{(M!)^{(m-1)m}} = \frac{p_{m-1,m,M}}{|\Psi_{m-1,m,M}|},$$

$$p_{m,M} \triangleq \sum_{\mathbf{P} \in \Psi_{m-1,M}} \text{perm} \begin{bmatrix} I & I & \cdots & I \\ I & P_{11} & \cdots & P_{1,m-1} \\ \vdots & \vdots & \cdots & \vdots \\ I & P_{m-1,1} & \cdots & P_{m-1,m-1} \end{bmatrix} \text{ and}$$

$$q_{m,M} \triangleq \frac{p_{m,M}}{(M!)^{(m-1)^2}} = \frac{p_{m,M}}{|\Psi_{m,M}|},$$

**Lemma 12.** We have the following relations between  $q_{m,M}$ ,  $q_{m-1,m,M}$ , and the average  $\langle \text{perm}(\theta^{\uparrow \mathbf{R}}) \rangle_{\mathbf{R} \in \Psi_{m,M}}$ :

$$q_{m,M} = q_{m-1,m,M} = \langle \text{perm}(\theta^{\uparrow \mathbf{R}}) \rangle_{\mathbf{R} \in \Psi_{m,M}}$$

*Proof:* Any  $\mathbf{R}$ -lifting of the matrix  $\theta$ ,  $\mathbf{R} = (R_{ij}) \in \Psi_{m,M}$ , can be written as

$$\theta^{\uparrow \mathbf{R}} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{bmatrix}.$$

Because the permanents are not affected by row or column permutations, we can apply permutations of columns to reduce the lifting matrix to the simpler form

$$\begin{bmatrix} I & I & \cdots & I \\ R_{21}R_{11}^T & R_{22}R_{12}^T & \cdots & R_{2m}R_{1m}^T \\ \vdots & \vdots & \cdots & \vdots \\ R_{m1}R_{11}^T & R_{m2}R_{12}^T & \cdots & R_{mm}R_{1m}^T \end{bmatrix} \triangleq \begin{bmatrix} I & I & \cdots & I \\ Q_{11} & Q_{12} & \cdots & Q_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ Q_{m-1,1} & Q_{m-1,2} & \cdots & Q_{m-1,m} \end{bmatrix}, \quad (3)$$

followed by row permutations to reduce it to

$$\begin{bmatrix} I & I & \cdots & I \\ I & Q_{11}^T Q_{12} & \cdots & Q_{11}^T Q_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ I & Q_{m-1,1}^T Q_{m-1,2} & \cdots & Q_{m-1,1}^T Q_{m-1,m} \end{bmatrix} \triangleq \begin{bmatrix} I & I & \cdots & I \\ I & P_{11} & \cdots & P_{1,m-1} \\ \vdots & \vdots & \cdots & \vdots \\ I & P_{m-1,1} & \cdots & P_{m-1,m-1} \end{bmatrix}, \quad (4)$$

where  $\mathbf{Q} = (Q_{ij}) \in \Psi_{m-1,m,M}$  and  $\mathbf{P} = (P_{ij}) \in \Psi_{m-1,M}$ . Therefore, the average  $\langle \text{perm}(\theta^{\uparrow \mathbf{R}}) \rangle_{\mathbf{R} \in \Psi_{m,M}}$  of the permanents of  $\theta^{\uparrow \mathbf{R}}$  over all matrices  $\mathbf{R} \in \Psi_{m,M}$  equals both  $q_{m,M}$  and  $q_{m-1,m,M}$ . ■

**Lemma 13.** Let  $m \geq 2$  and  $M \geq 1$ . Let  $\mathbf{Q} = (Q_{ij}) \in \Psi_{m-1,m,M}$  and let  $\{[M] \setminus \beta_1, \dots, [M] \setminus \beta_m\}$  be a partition of  $[M]$  (i.e.,  $m$  disjoint sets of column indices with union equal to  $[M]$ ). Then

$$\sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \\ j \in [m]}} \text{perm} \begin{bmatrix} Q_{11,\beta_1} & Q_{12,\beta_2} & \cdots & Q_{1m,\beta_m} \\ \vdots & \vdots & \cdots & \vdots \\ Q_{m-1,1,\beta_1} & Q_{m-1,2,\beta_2} & \cdots & Q_{m-1,m,\beta_m} \end{bmatrix} \leq \sum_{\substack{P_{ij} \in \mathcal{P}_M \\ i,j \in [m-1]}} \text{perm} \begin{bmatrix} P_{11} & \cdots & P_{1,m-1} \\ \vdots & & \vdots \\ P_{m-1,1} & \cdots & P_{m-1,m-1} \end{bmatrix}. \quad (5)$$

*Proof:* See Appendix A. ■

**Corollary 14.** Let  $q_{m,M} = \langle \text{perm}(\theta^{\uparrow \mathbf{R}}) \rangle_{\mathbf{R} \in \Psi_{m,M}}$ , for  $m \geq 1$ , as defined above. Then

$$q_{m,M} \leq m^M q_{m-1,M} \quad (6)$$

*Proof:* The permanent is computed by summing all products of entries such that each row and each column contribute to the product exactly once. The first  $M$  rows can contribute with 1s from the identity matrices on the top block row by taking the 1 entries from the columns indexed by  $[M] \setminus \beta_1$  of the first block of  $M$  columns,  $[M] \setminus \beta_2$  of the second block of  $M$  columns, such that  $[M] \setminus \beta_2 \subseteq [M] \setminus ([M] \setminus \beta_1) = \beta_1$ , and finishing with  $[M] \setminus \beta_m$  of the  $m$ th block of  $M$  columns. In any such choice of  $M$  entries of 1 from the first  $M$  rows,  $[M] \setminus \beta_1, [M] \setminus \beta_2, \dots, [M] \setminus \beta_m$  must be a partition of  $[M]$ .

Let  $|\beta_i| = r_i$ , then  $r_1 + \dots + r_m = (m-1)M$ . Note that the columns in the sets  $[M] \setminus \beta_1, [M] \setminus \beta_2, \dots, [M] \setminus \beta_m$  cannot contribute anymore to the rest of the product entries. There are  $\binom{M}{M-|\beta_1|} = \binom{M}{M-r_1} = \binom{M}{r_1}$  ways to choose the  $r_1$  columns from the first block of  $M$  columns,  $0 \leq r_1 \leq M$ . Then there are  $\binom{r_1}{r_2}$  ways to choose the  $r_2$  columns from the second block of  $M$  columns indexed by the remaining  $r_1 = |\beta_1|$  indices,  $0 \leq r_2 \leq r_1$ , etc., finishing with  $\binom{r_{m-2}}{r_{m-1}}$  ways to choose the  $r_{m-1}$  columns in the  $(m-1)$ th block of  $M$  columns,  $0 \leq r_{m-1} \leq r_{m-2}$ . The last  $M$  columns are uniquely determined by the fact that the corresponding rows in the top block should give a partition of  $[M]$ . Therefore, we can choose the 1s in the first  $M$  rows in exactly  $\sum_{r_1=0}^M \binom{M}{r_1} \sum_{r_2=0}^{r_1} \binom{r_1}{r_2} \dots \sum_{r_{m-1}=0}^{r_{m-2}} \binom{r_{m-2}}{r_{m-1}}$  ways, followed by products with entries in the next  $M+1, M+2, \dots, (m-1)M+1, (m-1)M+2, \dots, mM$  rows. Equivalently, we can compute and upper bound  $q_{m,M}$  in the following way.

$$\begin{aligned} q_{m,M} = q_{m-1,m,M} &= \frac{\sum_{r_1=0}^M \binom{M}{r_1} \sum_{r_2=0}^{r_1} \binom{r_1}{r_2} \dots \sum_{r_{m-1}=0}^{r_{m-2}} \binom{r_{m-2}}{r_{m-1}}}{(M!)^{(m-1)m}} \times \\ &\times \sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \\ j \in [m]}} \text{perm} \begin{bmatrix} Q_{11,\beta_1} & Q_{12,\beta_2} & \dots & Q_{1,m,\beta_m} \\ \vdots & \vdots & & \vdots \\ Q_{m-1,1,\beta_1} & Q_{m-1,2,\beta_2} & \dots & Q_{m-1,m,\beta_m} \end{bmatrix} \\ &\leq \sum_{r_1=0}^M \binom{M}{r_1} \sum_{r_2=0}^{r_1} \binom{r_1}{r_2} \dots \sum_{r_{m-1}=0}^{r_{m-2}} \binom{r_{m-2}}{r_{m-1}} q_{m-1,M} \\ &= q_{m-1,M} \sum_{r_1=0}^M \binom{M}{r_1} \sum_{r_2=0}^{r_1} \binom{r_1}{r_2} \dots \sum_{r_{m-1}=0}^{r_{m-2}} \binom{r_{m-2}}{r_{m-1}} \\ &= m^M q_{m-1,M}. \end{aligned}$$

The inequality above holds due to Lemma 13 and to the fact that  $\frac{1}{(M!)^{(m-1)m}} \leq \frac{1}{(M!)^{(m-1)^2}}$ . ■

**Theorem 15.** Let  $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{F}_2^{m \times (m+1)}$ .

Then, for all  $M \geq 1$ , its degree- $M$  Bethe perm-vectors  $\omega_{B,M}$  and its Bethe perm-vector  $\omega_B$ , based on  $\beta \triangleq [m+1]$ , are pseudocodewords for  $\mathbf{H}$ .

*Proof:* From Lemma 12 we know that  $\omega_{B,M} = (q_{m,M}^{1/M}, q_{m-1,M}^{1/M}, \dots, q_{m-1,M}^{1/M})$ , and from Cor. 14 we know the inequality  $q_{m,M}^{1/M} \leq m q_{m-1,M}^{1/M}$ . We obtain that  $\omega_{B,M}$  is a pseudocodeword. Taking the limit and applying Lemma 11 we get that  $\omega_B$  is also a pseudocodeword. ■

Note that Theorem 15 is the same as Theorem 10 stated and proved for the particular matrix  $\mathbf{H}$  displayed above.

We are now ready to prove Theorem 10. Since the components of a perm-pseudocodeword based on a set  $\beta$  of size  $m+1$  are defined as zero outside  $\beta$ , we can assume, without loss of generality, that  $n = m+1$  and prove the following version of Theorem 10.

**Theorem 16.** Let  $\mathbf{H}$  be any  $m \times (m+1)$  binary matrix. Then, for all  $M \geq 1$ , its degree- $M$  Bethe perm-vectors  $\omega_{B,M}$ , and its Bethe perm-vectors  $\omega_B$ , based on  $\beta = [m+1]$ , are pseudocodewords for  $\mathbf{H}$ .

*Proof:* See Appendix B for a sketch of the proof of this theorem in the general case. ■

#### IV. COMPUTATION OF BETHE PSEUDOCODEWORDS FOR $m=2$ AND $m=3$

##### A. Case $m=2$

In this case, we can compute the permanent average  $q_{2,M} = \langle \text{perm}(\theta^{\uparrow Q}) \rangle_{Q \in \Psi_{2,M}}$  exactly.

**Theorem 17.**  $q_{2,M} = M+1$ .

*Proof:* We want to compute  $\text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix}$ , for  $P \in \mathcal{P}_M$ . Since the permanent is computed by summing all products of entries such that each row and each column contributes to the product exactly once, the first  $M$  rows can contribute with 1s from the two adjacent identity matrices (on top) by choosing the 1 entries from the set of columns  $[M] \setminus \beta_1$  with  $r \triangleq |\beta_1|$ ,  $0 \leq r \leq M$ , from the first block, and  $\beta_1$  from the second block. This implies that the columns indexed by  $[M] \setminus \beta_1$  from the first block and  $\beta_1$  from the second block cannot contribute anymore to the rest of the product entries. Therefore, we obtain

$$\text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix} = \sum_{r=0}^M \binom{M}{r} \sum_{\substack{\beta_1 \subseteq [M] \\ |\text{supp}(\beta_1)| = r}} \text{perm} [I_{\beta_1} \ P_{[M] \setminus \beta_1}].$$

Then

$$\begin{aligned} q_{2,M} &= \frac{\sum_{P,Q,R,S \in \mathcal{P}_M} \text{perm} \begin{bmatrix} P & Q \\ R & S \end{bmatrix}}{(M!)^4} = \frac{\sum_{P \in \mathcal{P}_M} \text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix}}{M!} \\ &= \sum_{r=0}^M \binom{M}{r} \sum_{\substack{\beta_1 \subseteq [M] \\ |\text{supp}(\beta_1)| = r}} \sum_{P \in \mathcal{P}_M} \text{perm} [I_{\beta_1} \ P_{[M] \setminus \beta_1}] \\ &= \frac{\sum_{r=0}^M \binom{M}{r} r! (M-r)!}{M!} = M+1. \end{aligned}$$

This last equality holds because the permanent of the  $M \times M$  matrix  $[I_{\beta_1} \ P_{[M] \setminus \beta_1}]$  is 0 unless all its columns are different, or equivalently, unless the columns of  $I$  indexed by  $\beta_1$  are a permutation of the columns of  $P$  indexed by  $\beta_1$ , (or, equivalently,  $[I_{\beta_1} \ P_{[M] \setminus \beta_1}]$  is a permutation matrix) in which case the permanent is 1. There are  $(M-r)!$  ways of choosing the  $M-r$  columns of  $P$  and  $r!$  ways of choosing the remaining columns of  $P$  to obtain all possible matrices  $P \in \mathcal{P}_M$  that give a permanent 1. Given  $r$ , there are  $\binom{M}{M-r} = \binom{M}{r}$  ways of choosing the  $M-r$  columns in  $[M] \setminus \beta_1$ . ■

**Corollary 18.** The possible Bethe pseudocodewords for a  $2 \times 3$  matrix  $\mathbf{H}$  (with no zero row or column) are equivalent to the following vectors (see Def. 2).

- 1) If  $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ , then  $\omega_{B,M} \propto \omega_B = (1, 1, 1)$ .

- 2) If  $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ , then  $\omega_{B,M} = (1, 1, (M+1)^{1/M})$   
and  $\omega_B = (1, 1, 1)$ .  
3) If  $\mathbf{H} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ , then  $\omega_{B,M} = \omega_B = (1, 1, 1)$ .  
4) If  $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$  or  $\mathbf{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ , then  
 $\omega_{B,M} = \omega_B = (0, 1, 1)$ .

*Proof:* We apply Theorem 17 to compute the vectors in each case.

1)  $\omega_{B,M} = ((M+1)^{1/M}, (M+1)^{1/M}, (M+1)^{1/M}) \propto (1, 1, 1)$ . Taking the limit when  $M \rightarrow \infty$  we obtain

$$\omega_B = \lim_{M \rightarrow \infty} \omega_{B,M} = (1, 1, 1).$$

2)  $\omega_{B,M} = (t_{2,M}^{1/M}, t_{2,M}^{1/M}, q_{2,M}^{1/M}) = (1, 1, (M+1)^{1/M})$ , where  
 $t_{2,M} \triangleq \left\langle \text{perm} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{\uparrow \mathbf{P}} \right\rangle_{\mathbf{P} \in \Psi_{2,M}} = \text{perm} \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} = q_1 = 1$ .

Note that  $(M+1)^{1/M} \leq 2 \iff M+1 \leq 2^M$ , for all  $M \geq 1$ , proving that  $\omega_{B,M}$  is a pseudocodeword as the general theorem stated. It results in the Bethe perm-pseudocodeword  $(1, 1, 1)$ .

Cases 3) and 4) follow similarly.  $\blacksquare$

**Remark 19.** For all  $M \geq 1$ , the vector  $(1, 1, (M+1)^{1/M})$  is always a pseudocodeword for  $\mathbf{H}$  equal to the all-one  $2 \times 3$  matrix. Its AWGNC-pseudo-weight is equal to  $\frac{(2+(M+1)^{1/M})^2}{2+(M+1)^{2/M}}$  which is an increasing function that has a minimum equal to  $8/3$  and this is attained for  $M = 1$ , giving the pseudocodeword  $\omega_{B,1} = (1, 1, 2)$ . The Bethe perm-pseudocodeword  $\omega_B = (1, 1, 1)$  has AWGNC-pseudo-weight equal to 3.  $\square$

### B. Case $m = 3$

$$\text{Let } t_{3,M} \triangleq \left\langle \text{perm} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{\uparrow \mathbf{P}} \right\rangle_{\mathbf{P} \in \Psi_{3,M}}.$$

**Theorem 20.** The following is an exact formula for  $t_{3,M}$ .

$$t_{3,M} = \frac{\sum_{r=0}^M \binom{M}{r} \sum_{s=0}^r \binom{r}{s} (M-r+s)! (M-s)! \sum_{t=0}^{M-r} \binom{M-r}{t} (M-t)! (r+t)!}{M!^3}.$$

*Proof:* Applying Lemmas 31 and 32 in Appendix C, we can compute the permanent  $\text{perm} \begin{pmatrix} I & I & 0 \\ I & P & I \\ Q & R & I \end{pmatrix}$  as in (7) on top of the next page. The permanents

$$\text{perm} [I_{\gamma_1} \quad P_{\alpha_1 \setminus \gamma_1} \quad Q_{\gamma_2} \quad R_{[M] \setminus \alpha_1 \setminus \gamma_2}]$$

in (7) are 0 unless  $[I_{\gamma_1} \quad P_{\alpha_1 \setminus \gamma_1} \quad Q_{\gamma_2} \quad R_{[M] \setminus \alpha_1 \setminus \gamma_2}]$  is an  $M \times M$  permutation matrix. Then, taking the average permanent over all permutation matrices  $P, Q, R$ , yields

$$t_{3,M} = \frac{\sum_{r=0}^M \binom{M}{r} \sum_{s=0}^r \binom{r}{s} \sum_{t=0}^{M-r} \binom{M-r}{t} (M-r+s)! (M-s)! (M-t)! (r+t)!}{M!^3},$$

where  $\binom{0}{0} = 1$  by definition.  $\blacksquare$

**Remark 21.** The above formula for the average can also be rewritten as

$$t_{3,M} = \sum_{r=0}^M \frac{\sum_{s=0}^r \binom{M-r+s}{M-r} \binom{M-s}{M-r} \sum_{t=0}^{M-r} \binom{M-t}{r} \binom{r+t}{r}}{\binom{M}{r}^2}.$$

We obtain the following corollary.

**Corollary 22.** The Bethe pseudocodewords for two non-trivial  $3 \times 4$  matrices  $\mathbf{H}$  (with no zero row or column) given below are the following.

1) If  $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ , then

$$\omega_{B,M} = (t_{3,M}^{1/M}, (M+1)^{1/M}, (M+1)^{1/M}, (M+1)^{1/M}).$$

2) If  $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ , then

$$\omega_{B,M} = (q_{3,M}^{1/M}, (M+1)^{1/M}, (M+1)^{1/M}, (M+1)^{1/M}).$$

**Remark 23.** In the first case, observing that:

$$\begin{aligned} & \frac{\sum_{r=0}^M \binom{M}{r} \sum_{s=0}^r \binom{r}{s} (M-r+s)! (M-s)! \sum_{t=0}^{M-r} \binom{M-r}{t} (M-t)! (r+t)!}{M!^3} \\ & \leq \frac{\sum_{r=0}^M \binom{M}{r} \sum_{s=0}^r \binom{r}{s} (M-r)! M! \sum_{t=0}^{M-r} \binom{M-r}{t} M! r!}{M!^3} \\ & = \frac{\sum_{r=0}^M \binom{M}{r} (M-r)! M! M! r! \sum_{s=0}^r \binom{r}{s} \sum_{t=0}^{M-r} \binom{M-r}{t}}{M!^3} \\ & = \sum_{r=0}^M \sum_{s=0}^r \binom{r}{s} \sum_{t=0}^{M-r} \binom{M-r}{t} \sum_{r=0}^M 2^r \cdot 2^{M-r} \\ & = 2^M \sum_{r=0}^M 1 = 2^M (M+1), \end{aligned}$$

we see that indeed the vector  $\omega_{B,M}$  satisfies all the fundamental cone inequalities associated with the matrix  $\mathbf{H}$  giving that  $\omega_{B,M}$  is a pseudocodeword, as the Theorem 10 predicts. Taking the limit, we get that the Bethe perm-vector  $\omega_B$  is also a pseudocodeword.  $\square$

**Example 24.** We obtain, as expected, the pseudocodeword  $(4, 2, 2, 2) \propto (2, 1, 1, 1)$  for  $M = 1$  and, e.g.,  $(\sqrt{10}, \sqrt{3}, \sqrt{3}, \sqrt{3})$  for  $M = 2$  for the matrix  $\mathbf{H}$  in Cor. 22 1).  $\square$

**Remark 25.** Note that sometimes we can obtain valid pseudocodewords by taking block-perm-vectors based on a set of size  $m+1$  (without averaging over all possible lifts) as the following example shows.  $\square$

**Example 26.** Let  $M = 3$ ,  $P^s$ ,  $s = 0, 1, \dots, M-1$ , be the  $s$ -times cyclically left-shifted  $M \times M$  identity matrices and a matrix  $\mathbf{H}$  and its lifting  $\mathbf{H}^\uparrow$ :

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{H}^\uparrow = \begin{bmatrix} I & I & I & I \\ 0 & I & P & P^2 \\ 0 & I & P^2 & P \end{bmatrix}.$$

$$\begin{aligned}
& \text{perm} \begin{pmatrix} I & I & 0 \\ I & P & I \\ Q & R & I \end{pmatrix} = \text{perm}(I + P + Q + R) \\
&= \sum_{r=0}^M \sum_{\substack{\alpha_1 \subseteq [M] \\ |\text{supp}(\alpha_1)| = r}} \text{perm} [(I + P)_{\alpha_1} \quad (Q + R)_{[M] \setminus \alpha_1}] \\
&= \sum_{r=0}^M \sum_{\substack{\alpha_1 \subseteq [M] \\ |\text{supp}(\alpha_1)| = r}} \sum_{s=0}^r \sum_{\substack{\gamma_1 \subseteq \alpha_1 \\ |\text{supp}(\gamma_1)| = s}} \sum_{t=0}^{M-r} \sum_{\substack{\gamma_2 \subseteq [M] \setminus \alpha_1 \\ |\text{supp}(\gamma_2)| = t}} \text{perm} [I_{\gamma_1} \quad P_{\alpha_1 \setminus \gamma_1} \quad Q_{\gamma_2} \quad R_{[M] \setminus \alpha_1 \setminus \gamma_2}] \tag{7}
\end{aligned}$$

Let  $\mathbf{w}_M$  be computed as following

$$\begin{aligned}
\mathbf{w}_M &= \left( \text{perm}^{1/M} \begin{bmatrix} I & I & I \\ I & P & P^2 \\ I & P^2 & P \end{bmatrix}, \text{perm}^{1/M} \begin{bmatrix} P & P^2 \\ P^2 & P \end{bmatrix}, \right. \\
&\quad \left. \text{perm}^{1/M} \begin{bmatrix} I & P^2 \\ I & P \end{bmatrix}, \text{perm}^{1/M} \begin{bmatrix} I & P \\ I & P^2 \end{bmatrix} \right) = \\
&= (48^{1/M}, 2^{1/M}, 2^{1/M}, 2^{1/M}) = \sqrt[3]{2}(2\sqrt[3]{3}, 1, 1, 1).
\end{aligned}$$

This is a pseudocodeword equivalent to the pseudocodeword  $(2\sqrt[3]{3}, 1, 1, 1)$  with AWGNC-pseudo-weight 3.58.

Note that this is not always the case, an example can be easily found. In general, averages over all liftings need to be taken in order to obtain pseudocodewords.  $\square$

## V. CONCLUSIONS

In this paper we proved a stronger version of Vontobel's conjecture [7] on Bethe pseudocodewords and computed the families of pseudocodewords obtained by using the Bethe permanent and the degree- $M$  Bethe permanent in some particular cases. We also highlighted some interesting properties of Bethe permanents and their relationships to pseudocodewords.

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## APPENDIX

### A. Proof of Lemma 13

Let  $\beta \triangleq \{\beta_1, \dots, \beta_m\}$ , and let

$$T_\beta \triangleq \begin{bmatrix} Q_{11,\beta_1} & Q_{12,\beta_2} & \cdots & Q_{1m,\beta_m} \\ \vdots & \vdots & & \vdots \\ Q_{m-1,1,\beta_1} & Q_{m-1,2,\beta_2} & \cdots & Q_{m-1,m,\beta_m} \end{bmatrix}$$

like in the statement of Lemma 13. Note that  $T_\beta$  is of size  $(m-1)M \times (m-1)M$ . Each of the columns of  $[Q_{11,\beta_1} \quad Q_{12,\beta_2} \quad \cdots \quad Q_{1m,\beta_m}]$  has Hamming weight 1 and each of its rows has Hamming weight at most  $m$ . Let  $n_0, n_1, \dots, n_m$  be the number of rows of weight  $0, 1, \dots, m$ , respectively. The total number of rows is then  $n_0 + n_1 + n_2 + \cdots + n_m = M$  and the total number of entries equal to 1 in

the matrix is  $n_1 + 2n_2 + \cdots + mn_m = (m-1)M$ . We obtain:  $(m-1) \cdot (n_0 + n_1 + n_2 + \cdots + n_m) = n_1 + 2n_2 + \cdots + mn_m \Leftrightarrow (m-1)n_0 + (m-2)n_1 + (m-3)n_2 + \cdots + n_{m-2} = n_m$ . If  $n_m > 0$  we obtain that with each row of weight  $m$  there is at least one row of weight  $m-2$  or lower. If  $n_m = 0$  we obtain that  $n_0 = n_1 = \cdots = n_{m-2} = 0$  and hence all rows and columns have constant Hamming weight  $m-1$ . We will show that the first case can be reduced to the second case by a modification of the matrix that leaves the permanent unchanged or increases it.

We have two cases that we prove independently, the case of  $n_m > 0$  and the case of  $n_m = 0$ . The proof of the first case is followed by Example 27 and the proof of the second case contains Example 28 and is followed by Example 29 in order to better illustrate the mathematical techniques used in this proof.

### Case $n_m > 0$ .

*Proof:* We can assume that  $n_0 = 0$ , otherwise the permanent is 0 and does not contribute to the sum value of all permanents. There are  $m$  permutation matrices in the matrix  $[Q_{11} \quad Q_{12} \quad \cdots \quad Q_{1m}]$  from which columns indexed by  $\beta_1, \beta_2, \dots$ , and  $\beta_m$ , respectively, are chosen to give the  $M \times M(m-1)$  matrix  $[Q_{11,\beta_1} \quad Q_{12,\beta_2} \quad \cdots \quad Q_{1m,\beta_m}]$ . Therefore, the inequality  $n_m > 0$  means that there exists a column  $[0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0]^T$  that gets picked from each of the matrices  $Q_{11}, Q_{12}, \dots, Q_{1m}$ . This gives  $\beta_i > 0$ , for all  $i \in [m]$ . We can permute the first  $M$  rows of  $T_\beta$  (process that does not alter its permanent) and assume that this column is  $[1 \quad 0 \quad \cdots \quad 0]^T$ . We also obtain that there is another column which gets picked from at most  $m-2$  of the matrices  $Q_{11}, Q_{12}, \dots, Q_{1m}$ . We permute the first  $M$  rows of  $T_\beta$  by leaving the first row fixed, and assume that this column is  $[0 \quad 1 \quad 0 \quad \cdots \quad 0]^T$ . We can also assume that this column does not appear in  $Q_{11,\beta_1}$  and  $Q_{12,\beta_2}$ . Therefore, without loss of generality, we can assume that  $T_\beta$  is of the form displayed in (8), where, (up to column permutations within each block of  $M$  columns)

$$Q_{11,\beta_1} = \begin{bmatrix} 1 & & \\ \cdots & 0 & \cdots \\ & \vdots & \\ & 0 & \end{bmatrix}, \quad Q_{12,\beta_2} = \begin{bmatrix} 1 & & \\ \cdots & 0 & \cdots \\ & \vdots & \\ & 0 & \end{bmatrix},$$

[illegible]

$$T'_\beta = \left[ \begin{array}{c|c|c|c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \dots & \begin{matrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \hline \dots & \dots & \dots & \dots & \dots \\ \hline \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} & \dots & \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \\ \hline \begin{matrix} A_{\beta_1}^{(1)} & A_{\beta_2}^{(2)} & A_{\beta_3}^{(3)} & \dots & A_{\beta_m}^{(m)} \end{matrix} \right] \quad (9)$$

$$\text{perm}(T_\beta) - \text{perm}(T'_\beta) =$$

$$\text{perm} \left[ \begin{array}{c|c|c|c|c} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_1}^{(1)} \end{matrix} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_2}^{(2)} \end{matrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_3}^{(3)} \end{matrix} & \begin{matrix} 0 \\ 1 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_m}^{(m)} \end{matrix} \end{array} \right] - \text{perm} \left[ \begin{array}{c|c|c|c|c} \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_1}^{(1)} \end{matrix} & \begin{matrix} 0 \\ 1 \\ \vdots \\ 0 \\ A_{\beta_2}^{(2)} \end{matrix} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_3}^{(3)} \end{matrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \\ A_{\beta_m}^{(m)} \end{matrix} \end{array} \right] \quad (10)$$

$$\begin{aligned}
& \text{perm}(T_\beta) - \text{perm}(T'_\beta) \\
&= \text{perm} \left[ \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\beta_1}^{(1)} & A_{\beta_2}^{(2)} & A_{\beta_3}^{(3)} & & & A_{\beta_m}^{(m)} & \end{array} \right] - \\
& \text{perm} \left[ \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\beta_1}^{(1)} & A_{\beta_2}^{(2)} & A_{\beta_3}^{(3)} & & & A_{\beta_m}^{(m)} & \end{array} \right] \leqslant \quad (11) \\
& \text{perm} \left[ \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\beta_1}^{(1)} & A_{\beta_2}^{(2)} & A_{\beta_3}^{(3)} & & & A_{\beta_m}^{(m)} & \end{array} \right] - \\
& \text{perm} \left[ \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\beta_1}^{(1)} & A_{\beta_2}^{(2)} & A_{\beta_3}^{(3)} & & & A_{\beta_m}^{(m)} & \end{array} \right] \quad (12)
\end{aligned}$$

$$Q_{1j,\beta_j} = \begin{bmatrix} 1 & 0 \\ \cdots & 1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad j \geq 3,$$

and

$$\begin{bmatrix} A_{\beta_1}^{(1)} & A_{\beta_2}^{(2)} & \cdots & A_{\beta_m}^{(m)} \end{bmatrix} \triangleq \begin{bmatrix} Q_{21,\beta_1} & Q_{22,\beta_2} & \cdots & Q_{2m,\beta_m} \\ \vdots & \vdots & & \vdots \\ Q_{m-1,1,\beta_1} & Q_{m-1,2,\beta_2} & \cdots & Q_{m-1,m,\beta_m} \end{bmatrix}.$$

Let  $T'_\beta$  be the matrix displayed in (9). The two matrices  $T_\beta$  and  $T'_\beta$  differ in two positions only, in the first two rows and the second block of  $M$  columns. An entry equal to 1 in the first row gets “moved” to the position below in the second row and the same column. Therefore, the permanents of  $T_\beta$  and  $T'_\beta$  differ only in the elementary products (products containing exactly one entry from each row and each column) that contain the “moving 1 entry”; their difference is given in (10). The first matrix in (10) has its first row of weight 1 (changed from the first row of  $T_\beta$  of weight  $m$ ) and its second row of weight up to  $m-2$  (equal to the second row of  $T_\beta$ ). The second matrix in (10) has its first row of weight  $m-1$  (changed from the first row of  $T'_\beta$  of weight  $m$ ) and the second row of weight 1 (changed from the second row of  $T'_\beta$  of weight less than or equal to  $m-1$ ). Permuting the first two rows of the second matrix in (10) gives formula (11) and changing a zero entry into a 1 entry on the second row of the first matrix give a larger difference of permanents expressed in (12).

The last two matrices in (12) differ now only in the second row, namely, in the positions of the 1 entries on the second row in the last  $m-2$  blocks. They have both the first row of weight 1, the second row of weight  $m-1$ , and all the other rows equal to the corresponding rows of  $T_\beta$ . We now observe that allowing the matrices  $Q_{ij}$  with  $i \in [m-1], j \in [m], i \geq 2$ , to vary among all possible permutation matrices, while keeping the first  $M$  rows fixed, gives us two equal sets of matrices. This can be seen by interchanging two columns in each  $Q_{ij}$  of a matrix in the first set to obtain a permutation matrix  $Q'_{ij}$ , for all  $i \in [m-1], j \in [m], i \geq 2$  and  $j \geq 3$ . Setting also  $Q'_{ij} = Q_{ij}$ ,  $i \in [m-1], j \in [m], i \geq 2$  and  $j \in \{1, 2\}$ , then these choices of  $Q'_{ij}$  with  $i \geq 2$  and  $j \geq 1$  give a matrix in the second set, and vice-versa. The two sets being equal means that the sum of all differences of their permanents over all matrices  $Q_{ij}$  with  $i \in [m-1], j \in [m], i \geq 2$ , is 0. Since

$$\sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \setminus \{1\} \\ j \in [m]}} \text{perm}(T_\beta) - \text{perm}(T'_\beta) \leq 0$$

we obtain

$$\begin{aligned} \sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \\ j \in [m]}} \text{perm}(T_\beta) &= \sum_{\substack{Q_{1j} \in \mathcal{P}_M \\ j \in [m]}} \sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \setminus \{1\}}} \text{perm}(T_\beta) \\ &\leq \sum_{\substack{Q_{1j} \in \mathcal{P}_M \\ j \in [m]}} \sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \setminus \{1\}}} \text{perm}(T'_\beta) = \sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [m-1] \\ j \in [m]}} \text{perm}(T'_\beta) \end{aligned}$$

Therefore, by “moving” a 1 entry from a row of weight  $m$  (among the first  $M$  rows) to a row of weight at most  $m-2$  (see (8) and (9)) gives a new matrix  $T'_\beta$  of the same form as the original matrix, but with its first  $M$  rows with only  $n_m - 1$  rows of weight  $m$ . Summing over all possible permutation matrices, this change gives a larger value of the permanent sum, therefore we can work with this new matrix.

We proceed similarly with each block of  $M$  rows that contains a row of weight  $m$ , to obtain a matrix with each row of weight  $m-1$ . Thus, we substitute each matrix in the first sum in (5) having  $n_m > 0$  with a matrix that has each row of weight  $m-1$ . ■

**Example 27.** Let  $M = 3, m = 3$ ,

$$\mathbf{Q} \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and  $\beta_1 \triangleq \{1, 2\}, \beta_2 \triangleq \{1, 2\}, \beta_3 \triangleq \{1, 2\}$ . Then

$$T_\beta = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The second and the third rows of the first block of 3 rows of  $T_\beta$  have weight  $3 = m$  and weight  $1 = m-2$ , respectively. Similarly, the first and the second rows of the second block of 3 rows of  $T_\beta$  have weight  $3 = m$  and weight  $1 = m-2$ , respectively. We can modify the matrix in two steps, by constructing

$$T'_\beta = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

followed by constructing

$$T''_\beta = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and showing that the change from  $T_\beta$  into  $T'_\beta$  and then into  $T''_\beta$  only increases the average permanent over all permutation matrices  $\mathbf{Q} \in \Psi_{2,3,3}$ .  $T'_\beta$  has the first block of  $M = 3$  rows of weight  $2 = m-1$ , while  $T''_\beta$  has constant row weight  $2 = m-1$ . □



$$T_\beta \triangleq \begin{bmatrix} Q_{11,\beta_1} & Q_{12,\beta_2} & Q_{13,\beta_3} \\ Q_{21,\beta_1} & Q_{22,\beta_2} & Q_{23,\beta_3} \end{bmatrix} \triangleq \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (13)$$

$$\mathcal{S}_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathcal{S}_{13} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{S}_{22} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathcal{S}_{23} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (14)$$

$$P_{11} \triangleq [Q_{12,\beta_2} \quad \mathcal{S}_{12}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{12} \triangleq [Q_{13,\beta_3} \quad \mathcal{S}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$P_{21} \triangleq [Q_{22,\beta_2} \quad \mathcal{S}_{22}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{22} \triangleq [Q_{23,\beta_3} \quad \mathcal{S}_{23}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (15)$$

**Case  $n_m = 0$ .**

*Proof:* In this case, we can split each of the matrices  $Q_{i1,\beta_1}, i \in [m-1]$ , into  $m-1$  sets  $\mathcal{S}_{ij}$  of columns such that the matrices  $[Q_{ij,\beta_j} \quad \mathcal{S}_{ij}] \triangleq P_{i,j-1}$  are all permutation matrices of size  $M$ , for all  $i \in [m-1]$ , and  $j \in [m], j \geq 2$ . In other words, each matrix  $Q_{ij,\beta_j}$  with  $j \geq 2$  needs  $M - |\beta_j|$  columns to get completed to a permutation matrix  $P_{i,j-1}$ , and these columns may all be found in the matrix  $Q_{i1,\beta_1}, i \in [m-1]$ . Note that no one column is needed for completion by two different matrices  $Q_{ij,\beta_j}, Q_{ik,\beta_k}, j \neq k$ , because if so, then there would exist a row with a lower weight than  $m-1$ , thus contradicting the assumption we are under in this case.

Therefore, we can apply a permutation on the first columns in  $\beta_1$  of  $T_\beta$  to arrange them so that the first  $M - |\beta_2|$  columns of these form the set  $\mathcal{S}_{12}$  that completes the matrix  $Q_{12,\beta_2}$  to a permutation matrix  $P_{11}$ , the next  $M - |\beta_3|$  columns of these form the set  $\mathcal{S}_{13}$  that completes the matrix  $Q_{13,\beta_3}$  to a permutation matrix  $P_{12}$ , and so on. Since the permanent is not changed when permutations of columns or rows are applied, we can assume, without loss of generality, that  $T_\beta$  has this “order” on the first  $\beta_1$  columns. We denote by  $\sigma$  the permutation of columns of  $T_\beta$  required to change its first  $M$  rows as following (by extension, for simplicity,  $\sigma$  will also denote the induced permutation of columns on any submatrix of  $T_\beta$  obtained by erasing some of its rows):

$$\sigma [Q_{11,\beta_1} \quad \cdots \quad Q_{1,m-1,\beta_m}] = [P_{11} \quad \cdots \quad P_{1,m-1}].$$

Let

$$\sigma(T_\beta) = \begin{bmatrix} P_{11} & \cdots & P_{1,m-1} \\ R_{21} & \cdots & R_{2,m-1} \\ \vdots & \cdots & \vdots \\ R_{m-1,1} & \cdots & R_{m-1,m-1} \end{bmatrix},$$

where each  $R_{ij} = [Q_{ij,\beta_j} \quad \mathcal{S}_{1j}]$ ,  $j \geq 2$ , is a matrix of size  $M \times M$ . If  $R_{ij}$  are all permutation matrices, or equivalently,

if  $\mathcal{S}_{ij} = \mathcal{S}_{1j}$ , then

$$\sigma(T_\beta) = \begin{bmatrix} P_{11} & \cdots & P_{1,m-1} \\ P_{21} & \cdots & P_{2,m-1} \\ \vdots & \cdots & \vdots \\ P_{m-1,1} & \cdots & P_{m-1,m-1} \end{bmatrix}, \quad (16)$$

each  $P_{ij}$  is a permutation matrix of size  $M \times M$ , and  $T_\beta$  has the same permanent as the latter matrix.

**Example 28.** Let  $T_\beta$  with constant row weight  $m-1 = 2$  as in (13) and let  $\mathcal{S}_{12}, \mathcal{S}_{13}, \mathcal{S}_{22}, \mathcal{S}_{23}$ , as in (14). (Note that  $T_\beta$  is the matrix  $T''_\beta$  in Example 27.) Let

$$P_{11} \triangleq [Q_{12,\beta_2} \quad \mathcal{S}_{12}], \quad P_{12} \triangleq [Q_{13,\beta_3} \quad \mathcal{S}_{13}],$$

$$P_{21} \triangleq [Q_{22,\beta_2} \quad \mathcal{S}_{22}], \quad P_{22} \triangleq [Q_{23,\beta_3} \quad \mathcal{S}_{23}],$$

as described in (15)

Let  $\sigma$  denote the permutation operator that permutes the columns of  $T_\beta$  such that the first column gets moved to the 6th position and the second column gets moved to the 3rd position, thus completing the matrices  $Q_{12,\beta_2}$  and  $Q_{13,\beta_3}$  to the permutation matrices  $P_{11}$  and  $P_{12}$ , respectively. We compute  $\sigma(T_\beta)$  and obtain

$$\sigma(T_\beta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

since the  $\sigma$  column permutation completes also the matrices  $Q_{22,\beta_2}$  and  $Q_{23,\beta_3}$  to the permutation matrices  $P_{21}$  and  $P_{22}$ , respectively. This achieves the desired form.  $\square$

We will now address the case where the order does not get preserved in the next  $M$  rows. For simplicity, we suppose that  $\mathcal{S}_{2j} \neq \mathcal{S}_{1j}$ . Therefore, we need an extra permutation of

$$\begin{aligned}
& \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm}(T_\beta) - \text{perm}(T'_\beta) \\
&= \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \begin{bmatrix} Q_{11,\beta_1} & A_1 \\ Q_{21,\beta_1} & A_2 \\ \vdots & \vdots \\ Q_{m-1,1,\beta_1} & A_{m-1} \end{bmatrix} - \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \begin{bmatrix} Q_{11,\beta_1} & A_1 \\ Q'_{21,\beta_1} & A_2 \\ \vdots & \vdots \\ Q_{m-1,1,\beta_1} & A_{m-1} \end{bmatrix} \\
&= \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1} & A_1 & \\ 1 & 0 & \\ 0 & 1 & \\ \vdots & \vdots & \cdots \\ 0 & 0 & \end{array} \right] A_2 - \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1} & A_1 & \\ 0 & 1 & \\ 1 & 0 & \\ \vdots & \vdots & \cdots \\ 0 & 0 & \end{array} \right] A_2 \quad (17) \\
&= \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1 \setminus \{1\}} & A_1 & \\ 0 & & \\ 0 & & \\ \vdots & & \cdots \\ 0 & & \end{array} \right] A_2 \setminus \text{row1} + \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1 \setminus \{2\}} & A_1 & \\ 0 & & \\ 0 & & \\ \vdots & & \cdots \\ 0 & & \end{array} \right] A_2 \setminus \text{row2} \\
&\quad - \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1 \setminus \{1\}} & A_1 & \\ 0 & & \\ 0 & & \\ \vdots & & \cdots \\ 0 & & \end{array} \right] A_2 \setminus \text{row2} - \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1 \setminus \{2\}} & A_1 & \\ 0 & & \\ 0 & & \\ \vdots & & \cdots \\ 0 & & \end{array} \right] A_2 \setminus \text{row1} = 0 \quad (18) \\
&\quad - \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1 \setminus \{1\}} & A_1 & \\ 0 & & \\ 0 & & \\ \vdots & & \cdots \\ 0 & & \end{array} \right] A_2 \setminus \text{row1} - \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [m]}} \text{perm} \left[ \begin{array}{cc|c} Q_{11,\beta_1 \setminus \{2\}} & A_1 & \\ 0 & & \\ 0 & & \\ \vdots & & \cdots \\ 0 & & \end{array} \right] A_2 \setminus \text{row2}
\end{aligned}$$

columns within the block matrix  $Q_{21,\beta_1}$  so that we obtain the equality  $S_{2j} = S_{1j}$ . In addition, we need to show that this change does not alter the sum of the permanents of all  $\sigma(T_\beta)$  over all possible permutation matrices  $Q_{ij}$ . In fact, it is sufficient to show that the operation of interchanging two columns within the matrix  $Q_{21,\beta_1}$  does not change the sum of  $\text{perm}(T_\beta)$  when computed over all possible permutation matrices  $Q_{ij}$ . This fact, applied sequentially to each matrix  $Q_{i1,\beta_1}$ ,  $i \geq 2$ , until the desired form is obtained, will conclude our proof.

Without loss of generality, for simplicity, we can assume that the two columns of  $Q_{21,\beta_1}$  that get interchanged are the vectors  $[1 \ 0 \ 0 \ \cdots \ 0]^\top$  and  $[0 \ 1 \ 0 \ \cdots \ 0]^\top$ . Let us denote by  $Q'_{21,\beta_1}$  the matrix obtained from  $Q_{21,\beta_1}$ , and  $T'_\beta$  be the matrix obtained from  $T_\beta$ , after the two columns get interchanged. In equation (17) we compute the difference of the permanents of the two matrices and sum this differ-

ence over all permutation matrices  $Q_{1j}$  and  $Q_{2j}$ . We denote  $A_i \triangleq [Q_{i2,\beta_2} \ \cdots \ Q_{i,m-1,\beta_m}]$ , for all  $i \in [m-1]$ . After performing consecutive cofactor expansions on the first and second row of the second block of  $M$  rows of  $T_\beta$ , we obtain that this difference is given by (18), where  $A_2 \setminus \text{row1}$  denotes the submatrix of  $A_2$  obtained by erasing its first row and  $A_2 \setminus \text{row2}$  denotes the submatrix of  $A_2$  obtained by erasing its second row. We note that the first and the last sums of the four sums in (18) are equal, and so are the second and the third sums, and so the four sums cancel each other and give

$$\sum_{Q_{1j}, Q_{2j} \in \mathcal{P}_M} \text{perm}(T_\beta) - \text{perm}(T'_\beta) = 0.$$

Summing over all matrices  $Q_{ij} \in \mathcal{P}_M$ , we obtain

$$\sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i, j \in [M]}} \text{perm}(T_\beta) - \text{perm}(T'_\beta) =$$

$$= \sum_{\substack{Q_{ij} \in \mathcal{P}_M \\ i \in [M] \setminus \{1, 2\}}} \sum_{\substack{Q_{1j}, Q_{2j} \in \mathcal{P}_M \\ j \in [M]}} \text{perm}(T_\beta) - \text{perm}(T'_\beta) = 0.$$

Therefore we can change the position of the columns within each matrix  $Q_{i1, \beta_1}$ , for all  $i \geq 2$ , until we obtain a matrix of the form given in (16), without changing the sum of the permanents of  $T_\beta$  computed over all permutation matrices  $Q_{ij}$ . This, together with the previous case, concludes the proof. ■

**Example 29.** Let

$$T_\beta \triangleq \begin{bmatrix} Q_{11, \beta_1} & Q_{12, \beta_2} & Q_{13, \beta_3} \\ Q_{21, \beta_1} & Q_{22, \beta_2} & Q_{23, \beta_3} \end{bmatrix} \triangleq \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

with constant row weight  $m - 1 = 2$ . Let

$$S_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad S_{13} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$P_{11} \triangleq [Q_{12, \beta_2} \quad S_{12}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$P_{12} \triangleq [Q_{13, \beta_3} \quad S_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$P_{21} \triangleq [Q_{22, \beta_2} \quad S_{22}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P_{22} \triangleq [Q_{23, \beta_3} \quad S_{23}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let  $\sigma$  denote the permutation operator that permutes the columns of  $T_\beta$  such that the first column gets moved to the 6th position and the second column gets moved to the 3rd position, thus completing the matrices  $Q_{12, \beta_2}$  and  $Q_{13, \beta_3}$  to the permutation matrices  $P_{11}$  and  $P_{12}$ , respectively. We compute  $\sigma(T_\beta)$  and obtain

$$\sigma(T_\beta) = \begin{bmatrix} P_{11} & P_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The  $\sigma$  column permutation does not complete the matrices  $Q_{22, \beta_2}$  and  $Q_{23, \beta_3}$  to the permutation matrices  $P_{21}$  and  $P_{22}$ , respectively.

Let

$$T'_\beta \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

obtained by interchanging the 3rd and 6th columns of  $\sigma(T_\beta)$ . The above computations show that this change does not alter the average permutation over all permutation matrices  $\mathbf{Q} \in \Psi_{2,3,3}$ .  $T'_\beta$  has now the desired form. □

### B. Sketch of Proof of Theorem 16

*Proof:* We can modify Lemma 13 and its proof to include other types of matrices. For example, for a matrix  $\theta$  of size  $m \times m$  with  $r$  entries of 1 and  $m - r$  entries of zero on a row, we would modify the lemma to show that  $\text{perm}_{B,M}(\theta) \leq r \text{perm}_{B,M}(\hat{\theta})$ , where  $\hat{\theta}$  is an  $(m-1) \times (m-1)$  submatrix of  $\theta$  obtained by erasing the row of weight  $r$  and a column that gives the lowest nonzero permanent among these matrices  $(m-1) \times (m-1)$  matrices. The proof follows all the steps above. The case when  $\mathbf{H}$  has a column with weight 1 is basically the hardest one. In the other cases, we can use that  $\text{perm}_{B,M}(\mathbf{1}_{(m-1) \times (m-1)}) \leq \text{perm}_{B,M}(\theta)$  if  $\theta$  contains a submatrix  $\mathbf{1}_{(m-1) \times (m-1)}$  to obtain that the fundamental cone inequalities are satisfied for the row containing the only 1 of that column. The other inequalities are trivially satisfied (all vector components are equal) or can be reduced to the case of  $(m-1) \times m$  matrices. ■

### C. Necessary Lemmas

This Appendix contains a few results that we need in our computations. Versions of the first two lemmas can also be found in [17].

**Lemma 30.** Let  $c_i \in \mathbb{F}_2^{(M-1) \times 1}$ ,  $A \in \mathbb{F}_2^{(M-1) \times (M-2)}$  and the all-zero matrix  $0$  of size  $1 \times (M-2)$ . Then, the following “expansion” of the permanent holds:

$$\text{perm} \begin{bmatrix} 1 & 1 & 0 \\ c_1 & c_2 & A \end{bmatrix} = \text{perm} [c_1 + c_2 \quad A].$$

*Proof:* Applying the cofactor expansion on the first row, the above permanent can be rewritten, based on the row-linear property of both determinant and permanent, as

$$\text{perm} [c_2 \quad A] + \text{perm} [c_1 \quad A] = \text{perm} [c_1 + c_2 \quad A]. \quad \blacksquare$$

**Lemma 31.** Let  $A, B$  be two square matrices of the same size  $M$ . Then

$$\text{perm}(A + B) = \sum_{\alpha \subseteq [M]} \text{perm} [A_\alpha \quad B_{[M] \setminus \alpha}]. \quad (19)$$

*Proof:* The permanent of the matrix is a sum of all possible non-zero elementary products. Each elementary product contains a 1 entry from each row and each column and this entry can be taken either from the matrix  $A$  or  $B$ , giving the above sum. ■

Using these two lemmas the following result can be derived.

**Lemma 32.** *Let  $P, Q, R, S$  be permutation matrices, and  $I$  the identity matrix, all of size  $M$ . Then*

$$\text{perm} \begin{bmatrix} I & I & 0 \\ I & P & I \\ Q & R & I \end{bmatrix} = \text{perm}(I + P + Q + R). \quad (20)$$

*Proof:* We apply Lemma 30 sequentially for the first  $M$  rows and then the last  $M$  columns, and we get

$$\begin{aligned} \text{perm} \begin{bmatrix} I & I & 0 \\ I & P & I \\ Q & R & I \end{bmatrix} &= \text{perm} \begin{bmatrix} I + P & I \\ Q + R & I \end{bmatrix} \\ &= \text{perm}(I + P + Q + R). \quad \blacksquare \end{aligned}$$

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